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# STABILITY OF THE COMPRESSIBLE LAMINAR BOUNDARY LAYER\*

by

Lester Lees\*\* and Eli Reshotko\*\*\*

Guggenheim Aeronautical Laboratory  
California Institute of Technology

## ABSTRACT

In previous theoretical treatments of the stability of the compressible laminar boundary layer the effect of the temperature fluctuations on the "viscous" (rapidly-varying) disturbances is either ignored (Lees-Lin), or is accounted for incompletely (Dunn-Lin). A thorough reexamination of this problem shows that temperature fluctuations have a profound influence on both the "inviscid" (slowly-varying) and viscous disturbances above a Mach number of about 2.0. The present analysis includes the effect of temperature fluctuations on the viscosity and thermal conductivity, and also introduces the viscous dissipation term that was dropped in the earlier theoretical treatments.

Some important results of the present study are: (1) instead of being nearly constant across the boundary layer the amplitude of the inviscid pressure fluctuations decreases markedly with distance outward from the plate surface for Mach numbers greater than 3. This behavior means that the Reynolds shear stress near the critical layer is greatly reduced; (2) at Mach numbers less than about 2 dissipation effects are

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\*\* Professor of Aeronautics.

\*\*\* Graduate Student. Now at Lewis Research Center, NASA.

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minor, but they become extremely important at higher Mach numbers and tend to become the dominant effect for hypersonic Mach numbers.

(3) the rate of conversion of energy from the mean flow to the disturbance flow through the action of viscosity at the wall increases with Mach number.

(4) the minimum critical Reynolds number for insulated flat plate boundary layers decreases in the range  $0 \leq M \leq 3$ , and then rises very sharply for hypersonic Mach numbers.

Numerical examples illustrating the effects of compressibility (including neutral stability characteristics) are obtained at Mach numbers of 2.2 and 5.6. The calculated neutral stability diagrams are compared with the experimental results of Laufer and Vrebalovich at  $M = 2.2$ , and of Demetriades at  $M = 5.3$ .

### LIST OF SYMBOLS

	Dimensional Quantities	Dimensionless Quantities	Reference Quantities
Positional Coordinates:			
longitudinal	$x^*$	$x$	$\sqrt{\nu_e^* (x^*/u_e^*)}$
normal	$y^*$	$y$	$\sqrt{\nu_e^* (x^*/u_e^*)}$
Time	$t^*$	$t$	$1/(u_e^* \sqrt{\nu_e^* (x^*/u_e^*)})$
Velocity Components:			
longitudinal	$u^* = \bar{u}^* + u'^*$	$w(\eta) + f(\eta) e^{i\alpha(x-ct)}$	$u_e^*$
normal	$v^* = \bar{v}^* + v'^*$	$v(\eta) + \alpha \phi(\eta) e^{i\alpha(x-ct)}$	$u_e^*$
Density	$\rho^* = \bar{\rho}^* + \rho'^*$	$\rho(\eta) + r(\eta) e^{i\alpha(x-ct)}$	$\rho_e^*$
Pressure	$p^* = \bar{p}^* + p'^*$	$p(\eta) + \pi(\eta) e^{i\alpha(x-ct)}$	$p_e^*$
Temperature	$T^* = \bar{T}^* + T'^*$	$T(\eta) + \theta(\eta) e^{i\alpha(x-ct)}$	$T_e^*$
Viscosity Coefficient	$\mu^* = \bar{\mu}^* + \mu'^*$	$\mu(\eta) + m(\eta) e^{i\alpha(x-ct)}$	$\mu_e^*$
Thermal Conductivity	$k^* = \bar{k}^* + k'^*$	$\frac{\mu(\eta)}{\sigma} + \frac{m(\eta)}{\sigma} e^{i\alpha(x-ct)}$	$c_p \mu_e^*$

	Dimensional Quantities	Dimensionless Quantities	Reference Quantities
Wave Length	$\lambda^*$	$\lambda$	$\sqrt{\nu_*^* (x^*/u_*^*)}$
Wave Number	$\alpha^* = (2\pi/\lambda^*)$	$\alpha = (2\pi/\lambda)$	$\sqrt{u_*^* / \nu_*^* x^*}$
Disturbance Propagation Velocity	$c^*$	$c$	$u_*^*$

F	inviscid f function
G	$\pi'/a^2 \pi$
I	$\gamma-1$
K	$(\gamma-1) M_e^2 c^2$
L	$T_w$
L	length of plate
M <sub>e</sub>	local Mach number outside the mean boundary layer
M <sub>REL</sub>	$\frac{M_e (w-c)}{\gamma T}$ Mach number in wave coordinates
Q(x, y, t)	quantity of the total flow
$\bar{Q}(x, y)$	mean or steady component of flow quantity
Q'(x, y, t)	fluctuating component of flow quantity
q(y)	fluctuation amplitude
R*	gas constant
Re	Reynolds number based on reference length
Re <sub>0</sub>	Reynolds number based on momentum thickness
Re <sub>ref</sub>	Reynolds number based on reference length but with fluid properties evaluated at T <sub>ref</sub>
T <sub>0</sub>	stagnation temperature of external stream
T <sub>ref</sub>	reference temperature of mean boundary layer
a <sub>0</sub>	momentum thickness wave number $2\pi\theta/\lambda$
$\gamma$	ratio of specific heats
$\delta_w$	thickness of viscous layer near wall
$\epsilon$	small parameter $(aRe)^{-\frac{1}{2}}$
$\bar{\epsilon}$	small parameter $(aRe_{ref})^{-\frac{1}{2}}$ for present ordering
$\theta$	boundary layer momentum thickness
$\lambda$	wave length of disturbance
$\nu$	kinematic viscosity
$\sigma$	Prandtl number

$\tau$	Reynolds stress
$\Phi(y)$	inviscid $\phi$ function

#### Subscripts

c	quantity evaluated at critical point
e	local condition outside mean boundary layer (external)
i	imaginary part of quantity
inv	inviscid
n	quantity for neutral inviscid disturbance
v	viscous
w	quantity evaluated at the wall

A bar over a quantity indicates mean value.

Primes generally denote differentiation with respect to  $y$ . The few instances where primes denote a fluctuating quantity should not cause any confusion.

### 1. Introduction

Experimental results obtained by Laufer<sup>1\*</sup> and Vrebalovich at  $M_\infty = 2.2$  furnish definite proof of the existence of Tollmien waves in the supersonic laminar boundary layer on an insulated flat plate. The general shape of the stability diagram in the  $\alpha_0 - Re_0$  plane does not differ much from that found at low speeds; however, the minimum critical Reynolds number and the wave numbers and amplification rates of the self-excited disturbances are much lower. Tollmien waves were also found by Demetriades<sup>2</sup> in the laminar boundary layer of an insulated flat plate at  $M_\infty = 5.8$ . Here the minimum critical Reynolds number seems to be about one order of magnitude larger, while the characteristic wave numbers and amplification rates are even lower than at  $M_\infty = 2.2$ .

These experimental studies stimulated a reexamination of the whole theoretical basis of laminar stability at high Mach numbers. In the earlier treatments of Lin<sup>3,4</sup> and one of the present authors, it is tacitly assumed that the critical layer (wave speed = flow speed) lies close to the surface, so that the flow velocity with respect to the wave is small in the region between the surface and this layer. In that case the rate of change of density (or temperature) following a fluid particle is small for the "viscous"

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\* Superscripts refer to references listed at the end of the paper.

(rapidly-varying) disturbances, and these disturbances are treated as essentially incompressible. Dunn<sup>5</sup> and Lin pointed out that this conclusion is valid for subsonic or slightly supersonic speeds, but not for higher supersonic speeds, because the critical layer is no longer "close" to the surface. They obtained the "first-order" effect of temperature fluctuations on the viscous disturbances through the continuity equation, and they discussed the importance of the thermal boundary condition. However, only the so-called "leading terms" are retained in the energy and momentum equations. No numerical calculations utilizing this method are available.

At high Mach numbers the temperature fluctuations are dominant, and viscous dissipation must become important. In the present study we want to retain these effects, so far as possible. Even for the "inviscid" (slowly-varying) disturbances previous treatments<sup>3-5</sup> have employed approximate methods valid only up to moderate supersonic speeds, and for wave velocities not too close to free-stream speed. New methods must now be developed to cope with high supersonic and hypersonic speeds.

Only the simplest gas is considered here, namely one with constant specific heats, constant Prandtl number and constant chemical composition. Of course the temperature dependence of viscosity and thermal conductivity is taken into account. The laminar boundary layer is idealized as a plane, parallel flow, and each total flow quantity is regarded as being composed of a mean component which depends only on the distance normal to the plate surface, and a fluctuating component of infinitesimal magnitude. Thus

$$Q(x, y, t) = \bar{Q}(y) + Q'(x, y, t)$$

With these assumptions the coefficients of the linear partial differential disturbance equations are independent of both  $x$  and  $t$ , so that a disturbance of the form

$$Q'(x, y, t) = q(y) e^{i\alpha(x - ct)}$$

is suggested. Here  $q(y)$  is a complex amplitude and  $c$  is the complex phase velocity. The real part of  $c = c_r$  is the dimensionless velocity of wave propagation parallel to the surface, while  $\alpha c_i$  gives the amplification (or damping) rate.

The total flow must satisfy the same boundary conditions on the velocity as the original steady mean flow, so that the longitudinal and normal velocity fluctuations vanish at the surface, i. e.,  $f_w = 0$  and  $\phi_w = 0$ . In general the thermal boundary condition at the surface states that the instantaneous temperature and heat transfer rate are continuous across the solid-gas interface. However, most surface materials are so highly

conductive compared to gases that the temperature fluctuations at the surface are almost completely suppressed at the frequencies of interest for laminar stability. Therefore we take  $\theta_w = 0$ . In Section 2 we show that for disturbances propagating at subsonic velocities relative to the free stream all disturbance amplitudes vanish far from the plate surface, i. e.,  $q(y) \rightarrow 0$  as  $y \rightarrow \infty$ .

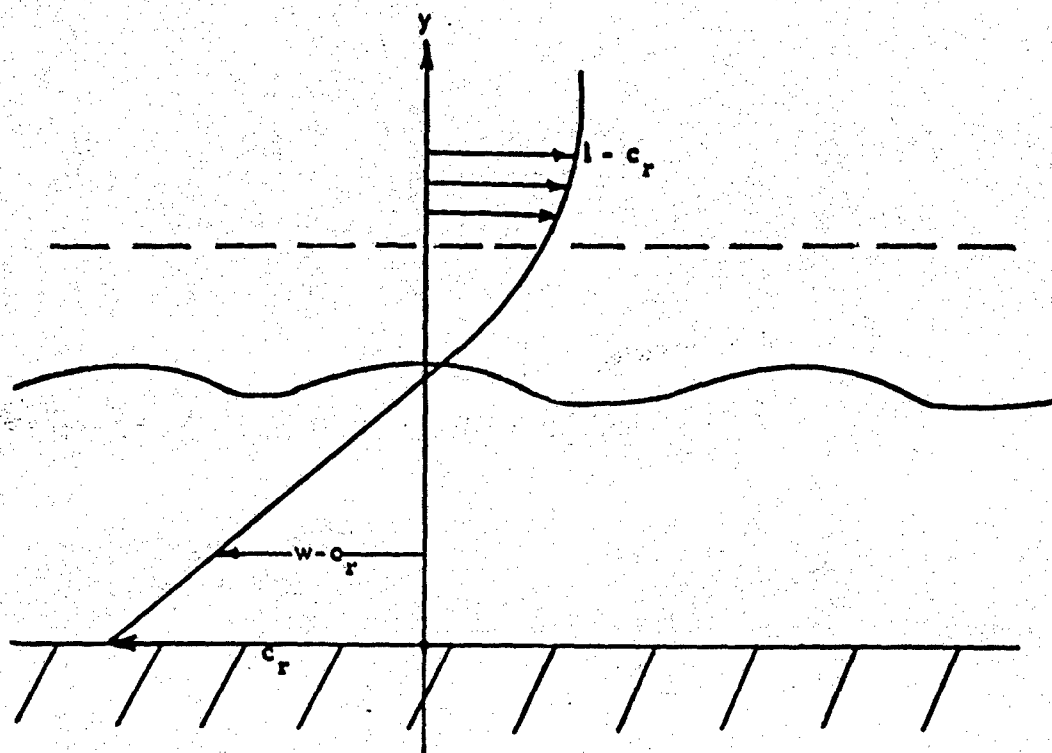
The purpose of the present paper is to bring out the physical mechanisms and main theoretical problems of laminar boundary layer stability at high Mach numbers. Details of the mathematical treatment (including the general thermal boundary condition  $a \theta_w' + b \theta_w = 0$ ) are contained in Reference 6. In Section 2 we delineate the principal disturbance flow parameters and regions in coordinates fixed in the wave. Section 3 is concerned with the mechanisms of production of disturbance energy at high speeds, and the conclusions to be drawn from the energy balance for neutral disturbances ( $c_i = 0$ ). In Section 4 some aspects of the eigenvalue problem are discussed, with particular emphasis on the role of the temperature fluctuations, and some numerical examples are presented for an insulated flat plate in air at  $M_\infty = 2.2$  and 5.6. Finally, in Section 5 we summarize our conclusions and the present state of our knowledge (and ignorance) of laminar stability at high Mach numbers.

## 2. Flow Regions and Parameters in Coordinates Fixed in the Wave

To an observer riding with the wave the entire flow field is steady. The uniform free stream is moving to the right with the velocity  $1 - c_r$ , while the plate surface is moving to the left with the velocity  $c_r$ . (See sketch on page 7.) The present discussion is concerned only with disturbances propagating at subsonic velocity relative to the free stream. In other words, the relative Mach number  $(M_{REL})_\infty = M_\infty (1 - c_r) \leq 1$ , or  $c_r \geq 1 - (1/M_\infty)$ . In such a steady, subsonic flow of unlimited extent it is well-known that all small disturbances die out with distance like  $e^{-\beta y}$  as  $y \rightarrow \infty$ , where

$$\beta = \alpha \sqrt{1 - (M_{REL})_\infty^2} = \alpha \sqrt{1 - M_\infty^2 (1 - c_r)^2}.$$

The restriction to subsonic relative motion evidently does not apply to the plate surface. In wave coordinates the plate Mach number is given by  $(M_{REL})_w = M_\infty c_r / \sqrt{T_w}$ . For an insulated surface and  $Pr = 1$ , one finds  $(M_\infty c_r) / (\sqrt{T_w}) \leq 1$  when  $M_\infty \lesssim 2.2$  (approximately, for  $\gamma = 1.4$ ). Thus



for  $M_e < 2.2$  the flow is everywhere subsonic with respect to the wave, but for  $M_e > 2.2$  a supersonic flow region exists near the plate surface. In fact, when  $M_e \gg 1$ ,  $c_r \rightarrow 1$ , and  $(M_e c_r) / (\sqrt{T_w}) \rightarrow \sqrt{5}$ . In this limiting case the sonic line defined by the relation  $M_e^2 (w - c)^2 = T$  occurs at  $w \approx 2/3^*$ .

The existence of an extensive supersonic flow region for sufficiently high  $M_e$  has a profound effect on the amplitude distribution for the "inviscid" pressure fluctuation between the plate surface and the critical layer ( $w = c$ ). At the plate surface  $(\phi_{INV})_w \approx 0$ ; therefore, in wave coordinates the amplitude of the pressure fluctuations near the plate surface satisfies the Prandtl-Glauert equation for a wavy disturbance in a steady, uniform flow, namely,

$$\pi'' + a^2 [M_{REL}^2 - 1] \pi = 0 \quad (1)$$

From Eq. (1) one sees that the amplitude of the pressure fluctuation decreases initially with distance away from the plate surface if  $M_{REL} > 1$ .

\* This sonic line does not introduce any real singularities into the linearized disturbance equations for a shear flow<sup>3</sup>.



and increases if  $M_{REL} < 1$ . If the y-coordinate is normalized by means of the Howarth-Dorodnitsyn transformation  $Y = \int_0^y (dy/T)$ , then Eq. (1) becomes\*

$$\frac{d^2 \pi}{dy^2} + \alpha^2 T_w^2 [(M_{REL})_w^2 - 1] \pi = 0 \quad (2)$$

In other words the proper parameter is not  $\alpha^2$ , but  $\alpha^2 T_w^2 \sim \alpha^2 M_\infty^4$  for an insulated plate. At subsonic or slightly supersonic free stream speeds  $\alpha^2 T_w^2 = O(\alpha^2)$ , and the increase in pressure amplitude between the plate surface and the critical layer is small. But at high supersonic and hypersonic speeds  $\alpha^2 T_w^2$  is no longer small, and  $(M_{REL})_w > 1$ , so that the decrease in pressure fluctuation amplitude outward to the critical layer is substantial. This phenomenon is not properly accounted for in previous theoretical treatments<sup>3-5</sup>, because the inviscid solutions are obtained by employing series expansions in powers of  $\alpha^2$ . Since  $\alpha^2$  is supposed to be small, it is tacitly assumed that  $|\pi_c/\pi_w|^2 = O(1)$ . The fact that  $|\pi_c/\pi_w|^2 \ll 1$  at high  $M_\infty$  has a strong influence on the Reynolds stress increment (or decrement) at the critical layer, and therefore on the energy balance for a neutral disturbance (Section 3).

The distinction between subsonic and supersonic flow regions in coordinates fixed in the wave makes sense only if the disturbances are largely "inviscid" (slowly-varying), as proposed originally by Prandtl. Viscosity and conductivity are important in two regions: (1) at the plate surface, where viscous solutions must be added to the inviscid solutions in order to satisfy the boundary conditions  $f_w = 0$ ,  $\theta_w = 0$ ; (2) at the "critical layer" ( $w = c$ ), where the longitudinal transport of vorticity and heat energy with respect to the wave vanishes, and the vertical transport of these quantities must be balanced by viscous diffusion and heat conduction. Clearly the proper local Reynolds number in coordinates fixed in the wave is  $\frac{\alpha Re}{\nu} |w - c_r|$ . At the plate surface the parameter  $(\frac{\alpha Re c}{\nu_w})^{-\frac{1}{2}}$  is a measure of the diffusion distance for vorticity during one period,\*\* so this inner boundary layer is thin when  $\alpha Re \gg 1$ . Similarly, the parameter

\* Since  $(\phi_{INV})_w \approx 0$ ,  $\pi'_w \approx 0$ .

\*\* The parameter  $(\frac{\alpha Re \sigma c}{\nu_w})^{-\frac{1}{2}}$  measures the corresponding diffusion distance for heat energy, where  $\sigma$  is the Prandtl number.

$$(\theta_{INV})_w = (\gamma - 1) M_o^2 (t_{INV}) \quad , \quad (6)$$

i. e., the "inviscid" temperature fluctuations normalized by the free stream temperature are of order  $M_o^2$ . Because of the boundary condition  $(\theta_{INV})_w + (\theta_v)_w = 0$ , the "viscous" temperature fluctuations are also of order  $M_o^2$ . Now the gradient of mean temperature is of order  $M_o^2$  and so is the whole level of mean temperature in the boundary layer when  $M_o^2 \gg 1$ . In other words the free stream static temperature is no longer relevant, and must be replaced by some representative temperature  $T_{REF} \sim M_o^2 T_o$ .

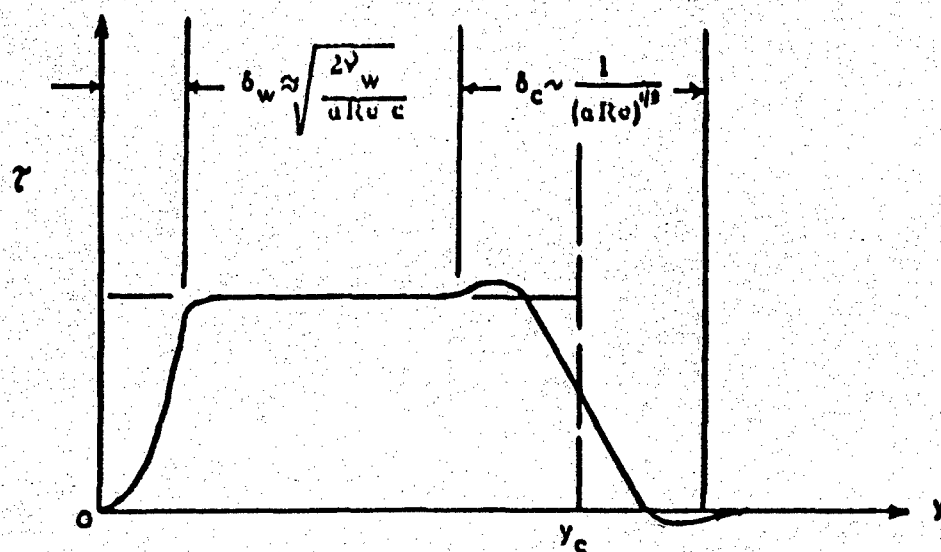
For example, the new parameter that orders the viscous terms in the equations of motion is given by  $\frac{1}{(\alpha Re_{REF})^{1/2}}$ , where  $Re_{REF} = \frac{Re}{\sqrt{\nu_{REF}}}$ ;

therefore,  $\bar{\epsilon} = (\nu_{REF})^{1/2} \epsilon$ , where  $\epsilon$  is the old parameter  $(\alpha Re)^{-1/2}$ . For a linear viscosity temperature relation  $\bar{\epsilon} \sim M_o^2$ . Terms in the equations of motion arising from the normal gradients of mean or fluctuating viscosity and conductivity, or from the viscous dissipation, were regarded as of order  $\epsilon$  in References 3, 4, and 5 compared with the "leading" viscous terms, but they are actually of order  $\bar{\epsilon}$ . At high supersonic or hypersonic free stream speeds these terms are likely to be of the same order as the so-called "leading terms" (Section 4).

### 3. Energy Balance and Reynolds Stress Distribution for a Neutral Disturbance

For a neutral disturbance the net rate of transfer of energy per cycle to the disturbance by the action of the Reynolds stress  $\gamma = -\rho u' \bar{v}'$  must be balanced exactly by the rate of dissipation of disturbance energy per cycle. As expected, the dissipation term is linear in the viscosity and is therefore of order  $1/\alpha Re$ . It is somewhat less obvious that the Reynolds stress also depends on viscosity because of the phase shift in disturbance velocities introduced near the plate surface. The Reynolds stress distribution across the boundary layer for  $\alpha Re \gg 1$  (Section 2) is sketched on page 11. [ The rate of transfer of energy to the disturbance is given by  $\int_0^\lambda \int_0^\infty \tau (du/dy) dy dx ]$ .

The Reynolds stress rises rapidly with distance away from the plate surface in a zone of thickness  $\delta_w \cong \sqrt{\frac{2\nu_w}{\alpha Re c}}$ , and then remains practically constant in the nearly-inviscid region between the surface and the critical



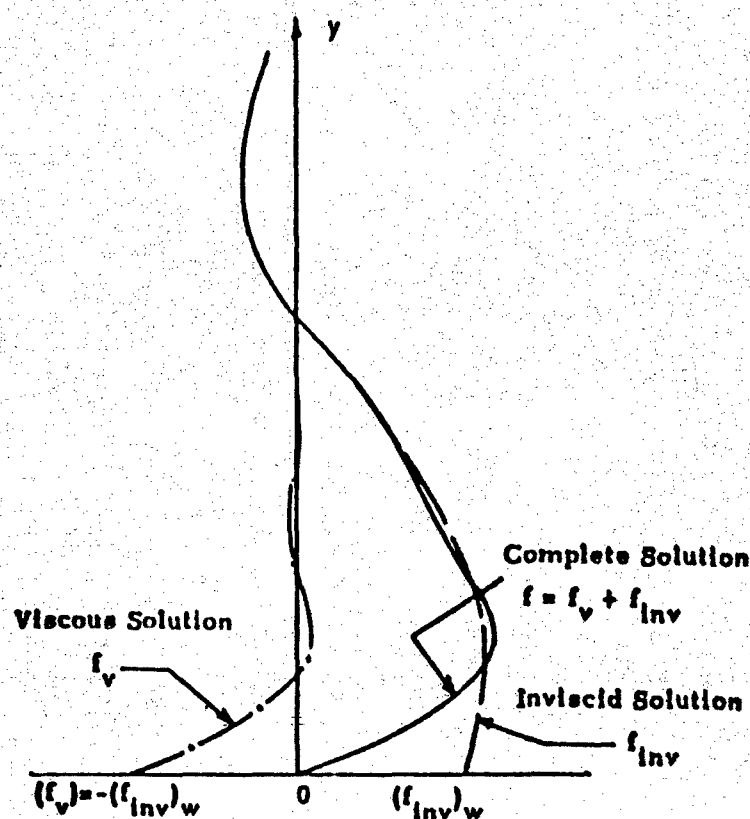
layer. Since  $\tau \rightarrow 0$  far from the surface,  $\tau$  must be cancelled by an equal and opposite jump in Reynolds stress at the critical layer. When  $\frac{aRe}{y_c} (1 - c) \gg 1$ , this jump can be calculated from the inviscid solutions (Section 2); viscosity and conductivity merely smooth out the transition. Thus

$$\Delta \tau \Big|_0^{\delta_w} = - \Delta \tau \Big|_{y_c - 0}^{y_c + 0} \quad (7)$$

Let us now examine each of these regions of Reynolds stress production (or destruction) separately.

#### (a) At the plate surface

For a fictitious inviscid, non-conducting gas, only the inviscid solutions apply. All disturbances vanish far from the surface, and the normal velocity fluctuation vanishes also at the plate surface. However, the inviscid temperature and longitudinal velocity fluctuations generally take on non-zero values at the surface. Now, for a real gas  $f_w = 0$  and  $\theta_w = 0$ , no matter how small the viscosity and thermal conductivity. Thus one must add viscous solutions determined by the conditions  $(f_v)_w = - (f_{INV})_w$  and  $(\theta_v)_w = - (\theta_{INV})_w$ ;  $\theta_v \rightarrow 0$  as  $y \rightarrow \infty$ . (See sketch on page 12.) By considering only the leading viscous terms in the momentum and energy equations in a thin layer near the plate surface, and recognizing that  $\phi$  and  $w \approx 0$  there, one finds the approximate viscous solutions



$$f_v = -(f_{INV})_w e^{-(1-i)(y/\delta_w)} \quad (8)$$

$$\theta_v = -( \theta_{INV} )_w e^{-(1-i)(y \sqrt{\sigma}/\delta_w)} \quad (9)$$

where  $\delta_w = \sqrt{\frac{2\nu_w}{\alpha Re c_r}}$ . But these additional temperature and longitudinal velocity fluctuations induce an increment in normal velocity fluctuation across the boundary layer. According to the equation of continuity and the equation of state

$$\frac{\partial v^*}{\partial y^*} = - \left[ \frac{\partial u^*}{\partial x^*} + \frac{1}{p^*} \frac{dp^*}{dt^*} - \frac{1}{T^*} \frac{dT^*}{dt^*} \right] \quad (10)$$

or

$$\phi_v' \approx -i f_v - (ic/T_w) \theta_v \quad (11)$$

By utilizing Eqs. (8) and (9) in Eq. (11), employing Eq. (6), and recognizing that  $\phi_v(\infty) = 0$ , one obtains

$$\int_0^{\infty} \phi_v' dy = (\phi_v)_w - \phi_v(\infty) = (\phi_v)_w$$

$$= - \frac{1(1+i)\delta_w}{2} (f_{INV})_w (1 + K\sigma^{-\frac{1}{2}}) \quad (12)$$

where

$$K = \frac{(\gamma - 1) M_\infty^2 c^2}{T_w} = (\gamma - 1) (M_{REL})_w^2 \quad (13)$$

But

$$\phi_w = (\phi_v)_w + (\phi_{INV})_w = 0$$

so that

$$(\phi_{INV})_w = -(\phi_v)_w \quad (14)$$

where  $(\phi_v)_w$  is given by Eq. (12), and the inviscid solutions must now be altered slightly to satisfy the boundary condition at the wall. The situation here is somewhat analogous to the streamline displacement effect produced in the external inviscid flow by the mean boundary layer.

It is precisely the fact that  $(\phi_{INV})_w \neq 0$  for  $\alpha Re$  large but not infinite that gives rise to the Reynolds stress associated with the inviscid solutions. Now  $\tau = -\rho \overline{u'v'}$ , so that

$$\tau = -(\rho/2) \alpha Rl (f\phi^*) \quad (15)$$

By employing Eqs. (12) - (15), one finds

$$\tau_{INV}(\delta_w) = \frac{\rho_w \alpha \delta_w}{4} \left| (f_{INV})_w \right|^2 (1 + K\sigma^{-\frac{1}{2}}) \quad (16)$$

[Note that  $\tau_{INV}$  is  $> 0$  and of order  $\frac{1}{\sqrt{\alpha Re}}$ . As the Mach number  $M_\infty$  increases  $K$  increases [Eq. (13)], and the Reynolds stress near the plate surface increases.]

#### (b) Across the critical layer ( $w = c$ )

The magnitude of the jump in Reynolds stress across the critical layer is given by the behavior of the inviscid solutions there<sup>3</sup>:

$$\Delta \tau \Big|_{y_c - 0}^{y_c + 0} = -(\alpha/2)(c/T_w w_w') \left| (f_{INV})_w \right|^2 v_o(c) \left| \frac{\pi_c}{\pi_w} \right|^2 \quad (17)$$

where

$$v_o(c) = -\pi \frac{w_w'}{T_w} \frac{c T_c^2}{(w_c')^3} \left[ (d/dy)(w'/T) \right]_{w=c}$$

[Eq. (5) connecting  $(f_{INV})_w$  and  $\pi_w$  has already been utilized.] In the limit  $\alpha Re \rightarrow \infty$ ,  $\tau_{INV}(\delta_w) \rightarrow 0$  [Eq. (16)], and the Reynolds stress vanishes everywhere. Thus the necessary (and sufficient) condition for the existence of a neutral inviscid disturbance is that  $(d/dy)(w'/T) = 0$  for some  $w \geq 1 - (1/M_0)$  (Reference 3). The value of  $w = c_g$  for which this condition is satisfied is shown in Figure 1. When  $c > c_g$ ,  $(d/dy)(w'/T) < 0$  and

$$\Delta \tau \int_{y_c - 0}^{y_c + 0} < 0, \text{ so that a neutral disturbance can in fact exist when } \alpha Re$$

is large but finite. Evidently the magnitude of this jump in  $\tau$  depends critically on the ratio  $|\pi_c/\pi_w|^2$ .

In Section 2 it was shown that the amplitude of the pressure fluctuation decreases with distance away from the plate surface at high supersonic speeds. A quantitative estimate of this decrease is obtained from the solutions of the inviscid equations for a neutral disturbance ( $c = c_g$ ). By employing Eqs. (3) and (10) one obtains

$$\phi = -if - \frac{i(w-c)\pi}{\delta} \quad (18)$$

and the momentum equations parallel and normal to the plate surface yield

$$i\rho(w-c)f + \rho w'\phi = -\frac{i\pi}{\gamma M_0^2} \quad (19)$$

$$i\alpha^2 \rho(w-c)\phi = -(\pi'/\gamma M_0^2) \quad (20)$$

By eliminating  $f$  and  $\phi$  one obtains a single differential equation for  $\pi$ . It turns out to be more convenient to introduce the Riccati transformation

$$G = (\pi'/\alpha^2 \pi) = (1/\alpha^2)(d/dy) \log \pi,$$

and the differential equation for  $G$  is (Reference 6)

$$G' = \left[ 1 - \frac{M_0^2 (w-c)^2}{T} \right] + \left( \frac{2w'}{w-c} - \frac{T'}{T} \right) G - \alpha^2 G^2. \quad (21)$$

At the plate surface  $G = 0$  when  $\alpha Re \rightarrow \infty$  and we obtain Eq. (1) of Section 2.

In Figure 2 the results of a number of numerical integrations of Eq. (21) are shown for an insulated plate surface, based on the mean flow profiles calculated by Mack<sup>7</sup>. The location of the critical layer coincides with the most outward zero of  $G$ . The behavior of the slope at  $y = 0$  follows the prediction made in Section 2. Since the area under each curve out to

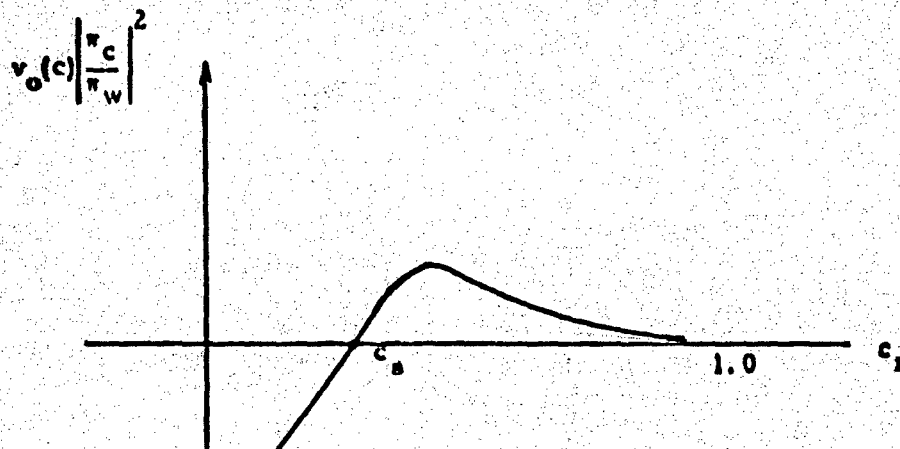
any distance from the surface is proportional to the logarithm of the pressure fluctuation amplitude at that location, it is clear that  $(\pi_c/\pi_w)^2$  decreases markedly with increasing Mach number at high  $M_0$ . In Figure 3 the ratio  $\pi_c/\pi_w$  is plotted as a function of  $M_0$ . Below  $M_0 \approx 2.5$ ,  $(\pi_c/\pi_w) = 0(1)$ . The integrations also yield the eigenvalues  $(\alpha_0)_s$ , and these values are shown in Figure 4. The value of  $(\alpha_0)_s$  is very small when  $M_0 \ll 1$ , increases in the subsonic range, and (after a dip near  $M_0 = 1.7$ ), reaches a maximum at about  $M_0 = 5.0$ . Beyond this Mach number  $(\alpha_0)_s$  approaches the asymptotic behavior  $(\alpha_0)_s \sim 1/M_0^2$ .

By referring to Eq. (17) and Figure 3, one sees that the stabilizing (or destabilizing) action of the jump in Reynolds stress at the critical layer decreases rapidly at high Mach number. According to Eqs. (7), (16), and (17) the condition for a neutral disturbance is

$$\sqrt{\alpha Re} = \sqrt{\frac{\gamma_w (w_w')^2}{2c^3}} \frac{(1 + K \sigma^{-\frac{1}{2}})}{v_0(c) \left| \pi_c/\pi_w \right|^2} \quad (22)$$

This relation is strictly applicable only when  $\alpha Re \gg 1$ , i. e., along the "upper branch" of the curve of neutral stability in the  $\alpha - Re$  plane. When  $M_0 < 2.5$ ,  $\left| \pi_c/\pi_w \right|^2 = 0(1)$ , and the principal effects of increasing Mach number are to shift  $c_r$  to higher values ( $c_r \approx 1 - (1/M_0)$ ), and to increase the rate of production of disturbance energy near the surface. As indicated by the factor  $(1 + K \sigma^{-\frac{1}{2}})$  in Eq. (22), this last effect shifts the eigenvalues along the upper branch to still larger values of  $\alpha Re$ , and is essentially destabilizing.

At hypersonic speeds, on the other hand, the behavior of the quantity  $v_0(c) \left| \pi_c/\pi_w \right|^2$  is the dominant factor. The variation of this quantity with  $c_r$  is indicated schematically in the accompanying sketch. Of course  $v_0(c_s) = 0$ , and the product  $v_0(c) \left| \pi_c/\pi_w \right|^2$  at first increases with increasing  $c_r > c_s$ . But  $\left| \pi_c/\pi_w \right|^2 \ll 1$  (Figure 3), and, moreover,  $\left| \pi_c/\pi_w \right|^2$  decreases with increasing  $c_r$ . Thus this product is always very small numerically, reaches a maximum at some value of  $c > c_s$ , and then decreases in value again as  $c_r \rightarrow 1$ . According to Eq. (22) this behavior means that  $\alpha Re$  decreases from infinity to some minimum (but very large) value, and then increases again. Thus the neutral stability curve in the  $\alpha - Re$  plane for  $\alpha Re \gg 1$  is a closed, isolated loop! Detailed studies bear out this unexpected conclusion (Section 4).



Since the absorption (or production) of energy near the critical layer becomes progressively less important with increasing  $M_\infty$ , viscous dissipation must become more important, in order to counterbalance the production of disturbance energy near the surface. But viscous dissipation  $\sim (1/\alpha Re)$ , so that the major stability problem occurs at lower values of  $\alpha Re$ , where the asymptotic expansion procedures hitherto employed may become inadequate. For an insulated surface  $K$  (Eq. (13)) approaches a limiting value of 2 for  $M_\infty^2 \gg 1$ . Thus  $\alpha Re$  at first decreases with increasing Mach number, and then possibly "levels out". In that case the rapid decrease in  $\alpha_0 \sim (1/M_\infty^2)$  would eventually insure a rapid increase in critical Reynolds number with increasing Mach number at hypersonic speeds.

#### 4. Eigenvalue Problem for Neutral Disturbances

Because of the increasing importance of viscous dissipation at high Mach numbers and the increasing relative magnitude of the temperature fluctuations, one is justifiably suspicious of ordering procedures based on the well-known parameter  $\epsilon \sim 1/(\alpha Re)^{1/2}$ . In fact the full linearized disturbance equations may be required. Nevertheless, in order to probe further into the nature of the high Mach number stability problem, we shall provisionally retain Prandtl's "splitting" of the solutions into "viscous" and "inviscid". However, we keep terms in the disturbance equations associated with normal gradients in fluctuating viscosity and conductivity, and with viscous dissipation, hitherto neglected. For example, in the energy equation the



heat conduction term  $(\partial/\partial y^*)(k^* \frac{\partial T^*}{\partial y^*})$  gives rise to three terms, involving successively the second and first normal derivatives of  $\theta$ , and  $\theta$  itself.

Order	Term	
1	$(\mu/\sigma) \theta''$	(the so-called "leading term" in the older analyses)
$\bar{\epsilon}$	$(2/\sigma)(d\mu/dT) \theta' T'$	(one contribution comes from the gradient in fluctuating conductivity, the other from the gradient in temperature fluctuation)
$\bar{\epsilon}^2$	$(\theta/\sigma)(d\mu/dT) T''$	(fluctuating conductivity)

In addition there is a term  $-\frac{\alpha^2 \theta}{\sigma}$  arising from longitudinal heat conduction  $(\partial/\partial x^*)(k^* \frac{\partial T^*}{\partial x^*})$ ; this term is of order  $\alpha^2 \bar{\epsilon}^2$  compared to  $\theta''$ . If terms of order  $\bar{\epsilon}^2$  are to be included, then to be consistent one should also include the  $\alpha^2 \bar{\epsilon}^2$  terms arising from streamwise gradients. In the present analysis only terms of order 1 and  $\bar{\epsilon}$  are retained; thus, this study amounts to a "first-order" investigation of the high Mach number effects. This procedure also has the important advantage that all viscous solutions are functions only of the single parameter  $\alpha Re$  (for a given flow), while the inviscid solutions are functions only of  $\alpha^2$ .

To this approximation the energy equation for the viscous disturbances is as follows:

$$\theta'' + (2/\mu)(d\mu/dT) T' \theta' + 2(\gamma-1)\sigma M_c^2 w' f' - \frac{i \alpha Re \sigma (w-c)}{\sqrt{\gamma}} \theta = \frac{\sigma \alpha Re T'}{\sqrt{\gamma}} \phi \quad (23)$$

In the Dunn-Lin<sup>5</sup> analysis only the first and last terms on the left-hand side of Eq. (23) appear, and the term containing the normal velocity fluctuation is also dropped because  $\phi$  is supposed to be of order  $\bar{\epsilon}$ . Similarly in the present study the equation of continuity for the viscous disturbances is

$$\phi' + i f - (T'/T) \phi - \frac{i(w-c)}{T} \theta = 0 \quad (24)$$

and the only term dropped is the pressure fluctuation term, which is of order  $\bar{\epsilon}^2$ . In the Dunn-Lin<sup>5</sup> analysis the  $\phi$  term is also dropped. In the Lees-Lin<sup>3,4</sup> study this term and the temperature fluctuation term do not appear, because the analysis is limited essentially to cases for which  $w - c = O(\bar{\epsilon})$ . In other words the viscous disturbances are essentially incompressible in the Lees-Lin analysis, and the viscous temperature fluctuations are irrelevant, so far as the eigenvalue problem is concerned.

In the present study the two momentum equations for the viscous disturbances including terms of order 1 and  $\bar{\epsilon}$  reduce to the following single equation:

$$f''' + \frac{2}{\mu} \frac{d}{dT} T' f'' + \frac{1}{\mu} \frac{d}{dT} w' \theta'' + \frac{1}{\gamma} \frac{\alpha Re (w-c)}{\gamma} \left[ \frac{T'}{T} f - f' - \frac{w'}{T} \theta \right] = 0 \quad (25)$$

In References 3-5 only the terms containing  $f'''$  and  $f'$  are retained.

Evidently the three equations [Eqs. (23) - (25)] are closely coupled and must be integrated simultaneously. Two linearly independent sets of solutions are distinguished by their behavior in the external flow, where  $w$  and  $T \rightarrow 1$ . One set is of the form  $f' \sim e^{-\sqrt{1/\alpha Re (1-c)} y}$ ,  $\theta = 0$  as  $y \rightarrow \infty$ ; the other set is of the form  $\theta \sim e^{-\sqrt{1/\alpha Re \sigma (1-c)} y}$ ,  $f = 0$ .\* [The third set corresponding to  $f' \rightarrow 0$  as  $y \rightarrow \infty$  corresponds to the inviscid solutions.] In order to satisfy the outer boundary conditions only the negative exponents are retained. The required numerical integrations for  $M_c = 2.2, 3.2$ , and  $5.6$  were performed on the Datatron 205 of the Caltech Computing Center using a Runge-Kutta integration method.\*\*

The boundary conditions at the plate surface are that  $f$ ,  $\phi$ , and  $\theta$  all vanish. These conditions lead to the complex secular equation for the eigenvalues, which after some manipulation, can be written in the form

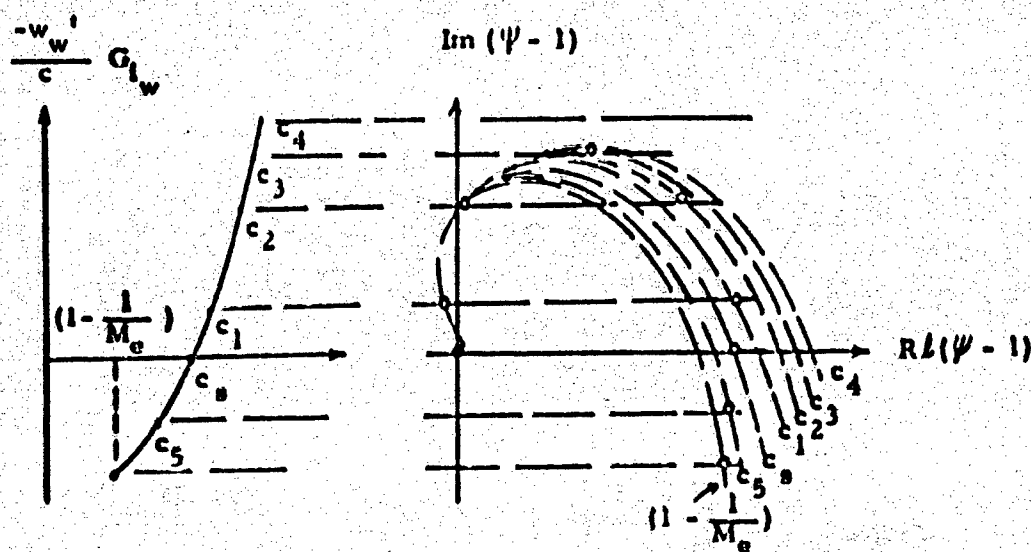
$$G_w = (c/w_w') (1 - \Psi) \quad (26)$$

where  $G_w = (\pi'/\alpha^2 \pi)_w$  and  $\Psi$  is an algebraic function of the values of the viscous solutions at  $y = 0$ .\* Thus  $G_w = G_w(\alpha; c, M_c^2)$  and  $\Psi = \Psi(\alpha Re; c, M_c^2)$ . For every value of  $c$  the real and imaginary parts of  $G(w_w'/c)$  and  $(1 - \Psi)$  are plotted in the complex plane, and the intersections determine the eigenvalues.

Suppose we consider first the simpler situation at  $M_c = 2.2$  and limit the discussion to neutral disturbances. Now  $-(w_w'/c)(G_1)_w$  depends mainly on  $v_o(c) |\pi_c/\pi_w|^2$ , and is practically independent of  $\alpha$ . In fact, it is closely related to the jump in Reynolds stress across the critical layer (Section 3). As shown in the sketch on page 19, there are two solutions for each value of  $-(w_w'/c)(G_1/w) > 0$ . At  $c = c_g$  this quantity vanishes, and one obtains the inviscid disturbances  $\alpha = \alpha_g$ ,  $\alpha Re \rightarrow \infty$ ; there is also

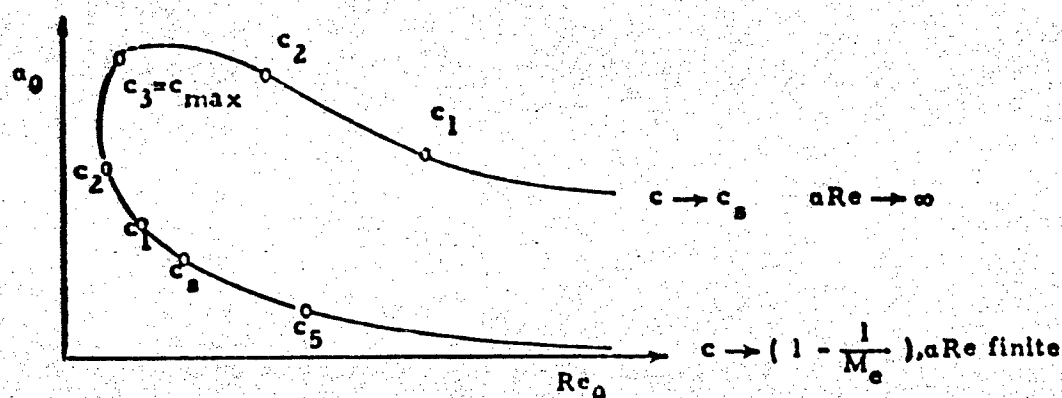
\* For details the reader is referred to Reference 6.

\*\* The authors are grateful to Mr. Kenneth Lock for programming and performing all high-speed digital computer calculations.



Imaginary Part of Inviscid Solutions

Viscous Solutions

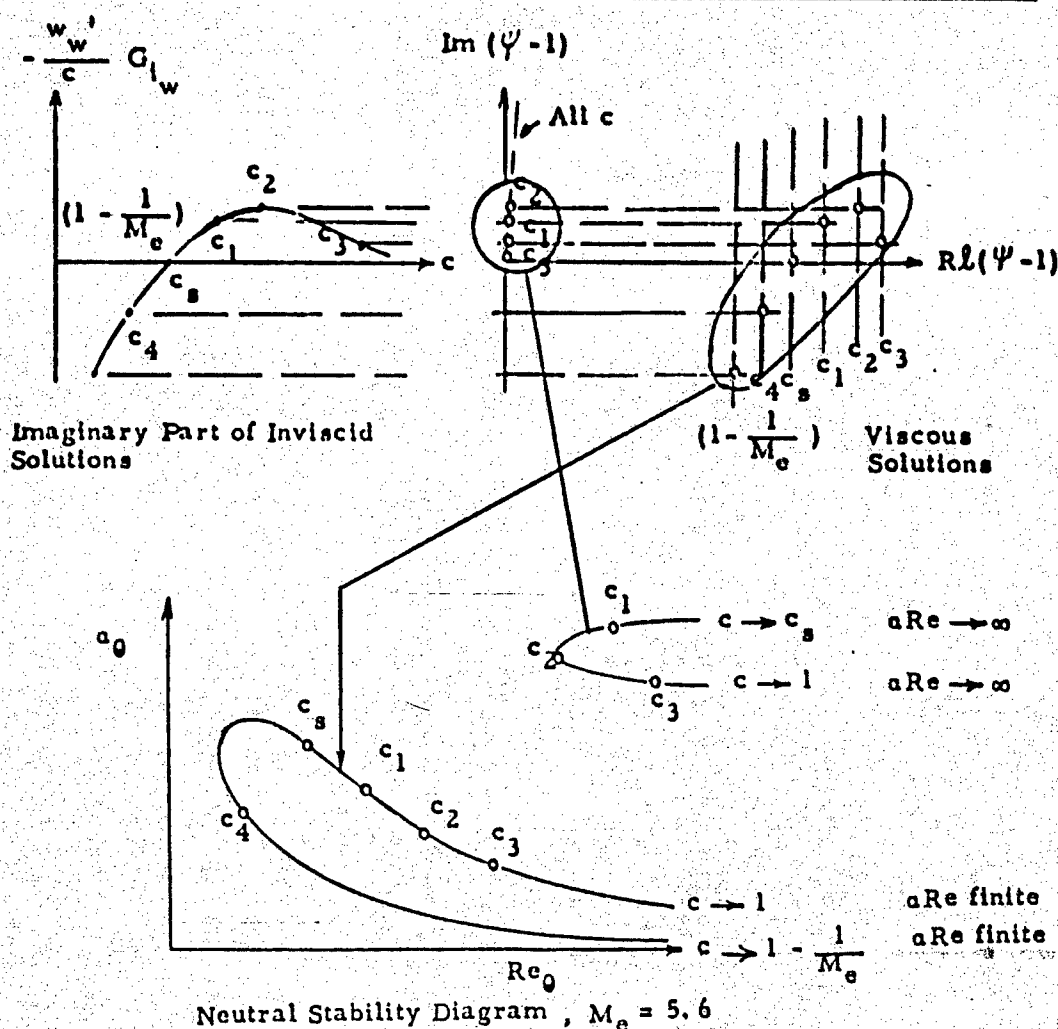
Neutral Stability Diagram ,  $M_e = 2.2$ 

a point on the "lower branch" of the stability diagram for  $c = c_s$ . As  $c_r$  increases above  $c_s$ , the quantity  $-(w'_w/c)(G_I)_w$  increases monotonically, and the eigenvalues obtained move toward each other on the upper and lower branches. Finally a value  $c = c_{MAX}$  is reached for which only one solution is obtained, and beyond which no solutions exist. When  $c < c_s$  only one solution is obtained and the rest of the lower branch is traced out. Along this branch  $\alpha_0 \rightarrow 0$  and  $c \rightarrow 1 - (1/M_e)$  as  $Re \rightarrow \infty$ , but  $\alpha Re$  is finite.<sup>3</sup>

In Figure 5 the results of the numerical calculations are compared with the Laufer<sup>1</sup>-Vrebalovich data and the Lees-Lin<sup>3,4</sup> theory. Along the upper branch ( $\alpha Re$  large) the present calculation agrees more closely with

the data (and with the Dunn-Lin<sup>5</sup> calculation) than the Lees-Lin calculation. Along the lower branch, however, the values of  $\alpha Re$  obtained by all three methods are somewhat too low, but the present method gives values of  $\alpha Re$  about 30 per cent higher than the Dunn-Lin method. Thus the inclusion of the terms of order  $\bar{E}$  seems to be a step in the right direction. At one calculated point on the lower branch, where  $\alpha_0 = 0.030$  and  $Re_0 = 89$ , one finds that  $\bar{E} = 0.54$ , so that even more terms are probably required.

Figure 6 shows the calculated amplitude distributions for one eigen-oscillation on the upper branch, and Figure 7 shows the mass flow and total temperature fluctuations. The agreement is surprisingly good, and confirms our belief that the laminar stability problem at Mach numbers below about 2.5 is now fairly well understood.



At  $M_0 = 5.6$  the situation is entirely different, as indicated already in the discussion of the energy balance [Eq. (22), Section 3]. Again  $-(w_w'/c)(G_1)_w$  vanishes at  $c = c_g$ , yielding two solutions, one corresponding to  $\alpha = \alpha_g$  and  $\alpha Re \rightarrow \infty$ , and the other to a finite value of  $\alpha Re$ . As  $c_r$  increases beyond  $c_g$  the quantity  $-(w_w'/c)(G_1)_w$  increases, but remains very small, because of the factor  $|\pi_c/\pi_w|^2$ . The upper isolated loop in the  $\alpha_0 - Re_0$  plane represents the solutions for  $\alpha Re \gg 1$  in the region  $c_g \leq c \leq 1$ . The corresponding second solutions for  $c_r = c_1, c_2, c_3$  trace out a new "upper loop", along which  $\alpha_0$  apparently  $\rightarrow 0$  as  $Re \rightarrow \infty$  and  $\alpha Re \rightarrow$  finite value. Presumably the single solutions obtained for  $1 - (1/M_0) < c < c_g$  trace out the rest of the lower loop.

The two loop behavior is shown in Figure 8 and compared with Demetriades' data<sup>2</sup>. Since the minimum value of  $Re_0$  on the upper loop is about  $10^5$  this loop is of little practical significance. The portion of the lower loop drawn in an unbroken curve has about the same shape as Demetriades' data, but the points lie about an order of magnitude lower in  $Re_0$ . However, the calculated points are themselves about one order of magnitude higher than those obtained using either the Lees-Lin or Dunn-Lin methods. The value of  $\bar{\epsilon}$  at the test point indicated in Figure 8 is about 2.0, which is some indication of the inadequacy of the ordering procedure based on  $\bar{\epsilon}$ .

Along the dotted portion of the lower loop in Figure 8,  $c \rightarrow 1$ , and  $\frac{\alpha Re}{\gamma_c}(1 - c)$  may be not be sufficiently large to justify the procedure of splitting the solutions into "inviscid" and "viscous" (See Section 2.).

Numerical results were also obtained for the "transitional" case  $M_0 = 3.2$ , but these results are not yet well enough understood to be discussed here.

## 5. Conclusions and Future Work

The present study of the stability of the compressible laminar boundary layer shows that the relative importance of the various physical mechanisms governing stability changes drastically at high supersonic Mach numbers.

1. Instead of being nearly constant across the boundary layer, the amplitude of the inviscid pressure fluctuations decreases markedly with

distance from the plate surface at Mach numbers greater than 3. Because of this behavior the rate of absorption (or production) of disturbance energy near the critical layer is greatly reduced, as compared with subsonic or slightly supersonic flows.

2. At the same time the rate of production of disturbance energy near the surface caused by the viscous phase shifts increases with Mach number.

3. Viscous dissipation becomes extremely important at high Mach number, since it must compensate for the effects mentioned in (1) and (2). This phenomenon is also foreshadowed by the increase in the relative magnitude of the temperature fluctuations. Accordingly, terms in the equations of motion involving gradients of viscosity or conductivity fluctuation, or viscous dissipation, which are neglected in the older analyses, must be included at high Mach numbers.

4. For free stream Mach numbers of 2.2, and below, only a single stability loop in the  $\alpha$  -  $Re$  diagram is obtained. Calculated neutral stability characteristics and disturbance amplitude distributions at  $M_\infty = 2.2$  are in good agreement with the Laufer<sup>1</sup>-Vrebalovich data.

5. At  $M_\infty = 5.6$  two distinct stability loops are obtained, but the minimum Reynolds number ( $Re_0$ ) for the upper loop is so high ( $\approx 10^5$ ) that it does not have much practical significance. The other loop is qualitatively similar to the experimental results of Demetriades<sup>2</sup> at  $M_\infty = 5.6$ . However, the calculated Reynolds numbers are still an order of magnitude lower than the experimental values, although they are in turn an order of magnitude larger than the values obtained from the Lees-Lin<sup>3,4</sup> or Dunn-Lin<sup>5</sup> methods.

6. At Mach numbers around 3.5 one obtains a transitional stability diagram between the "almost incompressible" behavior for  $M_\infty \leq 2.5$  and the hypersonic behavior for  $M_\infty \geq 5.0$ . This regime requires additional theoretical study.

7. The structure and solutions of the linearized disturbance equations must be carefully examined for the case  $c_r \rightarrow 1$ . In addition there is some question concerning the existence of multiple eigenvalues of the wave number for a neutral inviscid disturbance when the relative velocity between the wave and the plate surface is supersonic.

8. Asymptotic methods utilized in all boundary layer stability analyses, based on a "small" parameter of the form  $\epsilon = (\alpha Re)^{-1/2}$ , are no longer adequate at high Mach numbers. In fact the procedure of splitting the solutions into "viscous" (rapidly-varying), and inviscid ("slowly-varying"), is no longer justified. It is suggested that the complete, linearized disturbance equations should be integrated by methods similar to those developed in the present study for the separate viscous and inviscid solutions.

9. Some evidence exists that  $\alpha Re$  at first decreases with increasing Mach number, and then approaches a constant as viscous dissipation builds up. Since the wave number behaves asymptotically like  $(M_\infty^2)^{-1}$ , the minimum critical Reynolds number is likely to increase sharply for hypersonic speeds.

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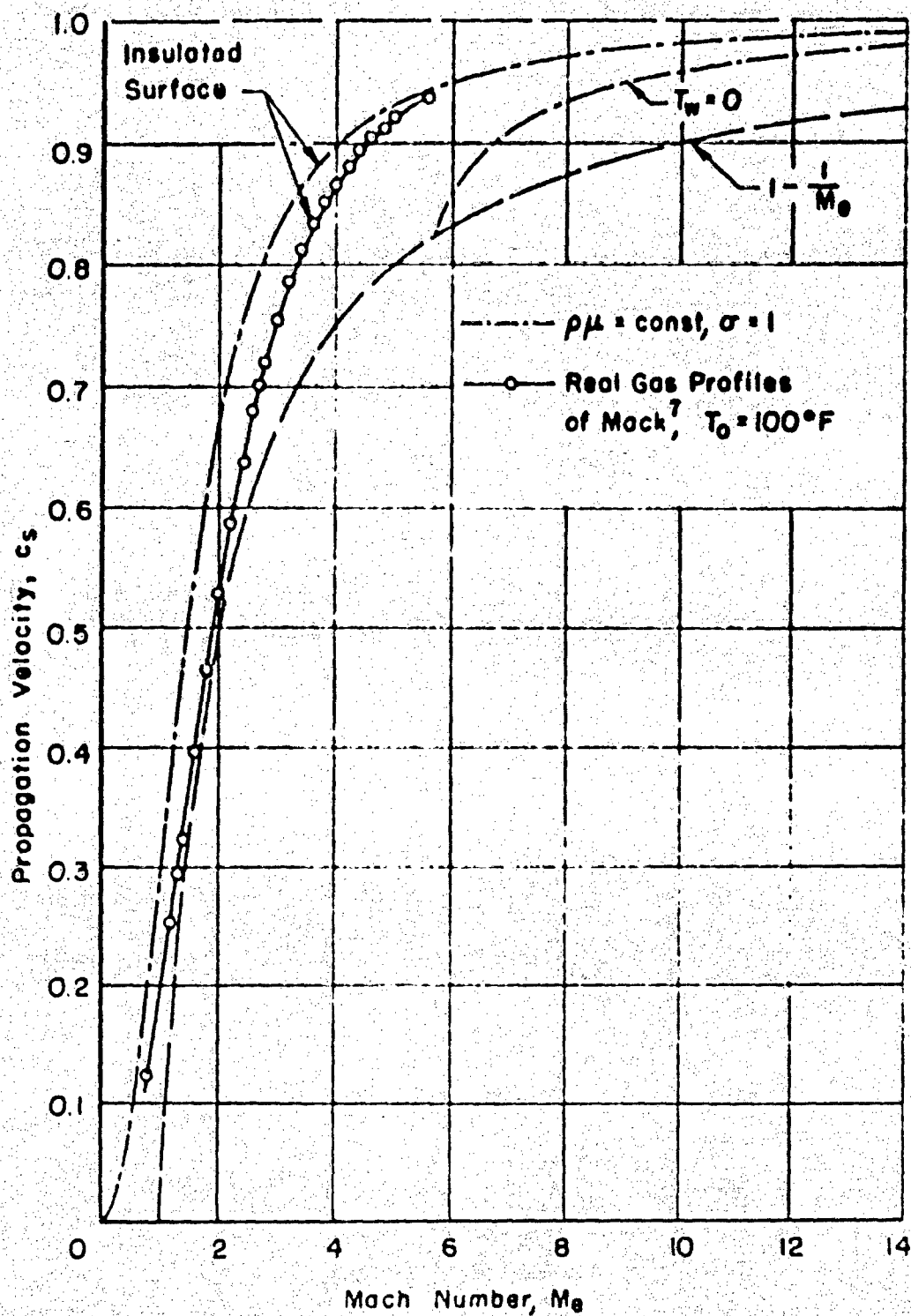
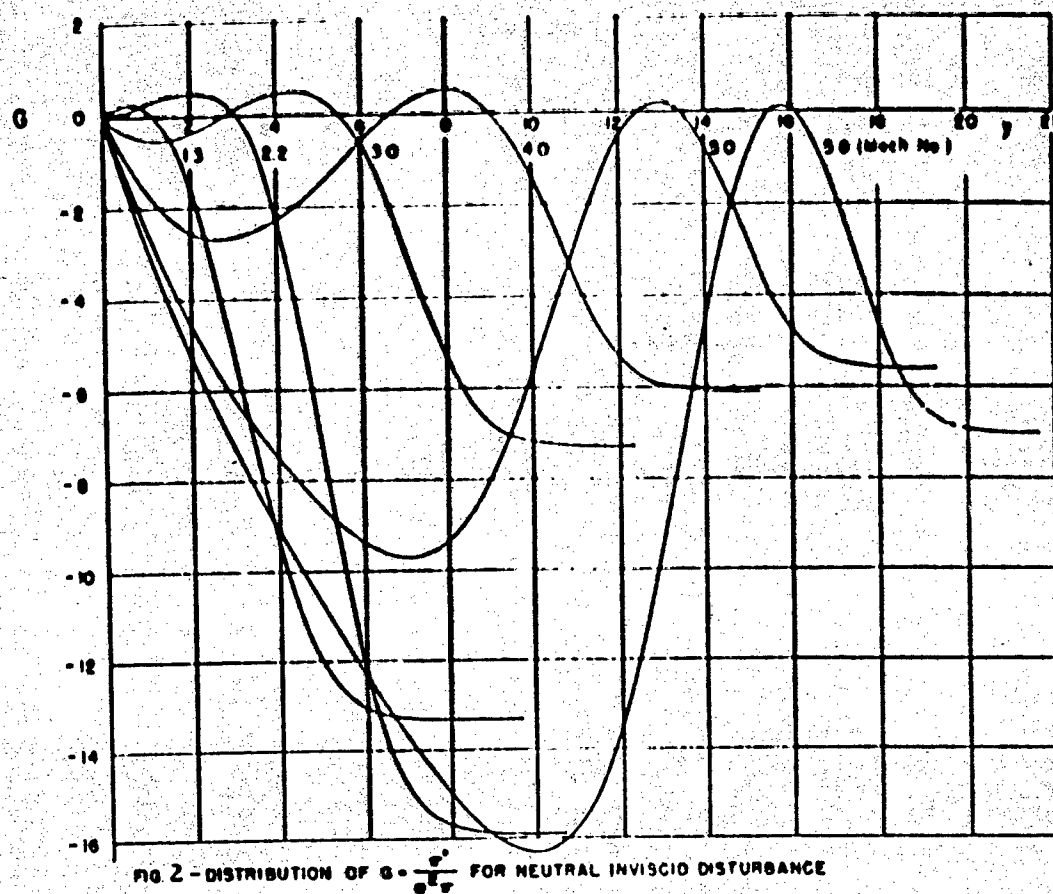


FIG. 1 - PROPAGATION VELOCITY OF NEUTRAL INVISCID DISTURBANCES FOR FLAT PLATE BOUNDARY LAYERS





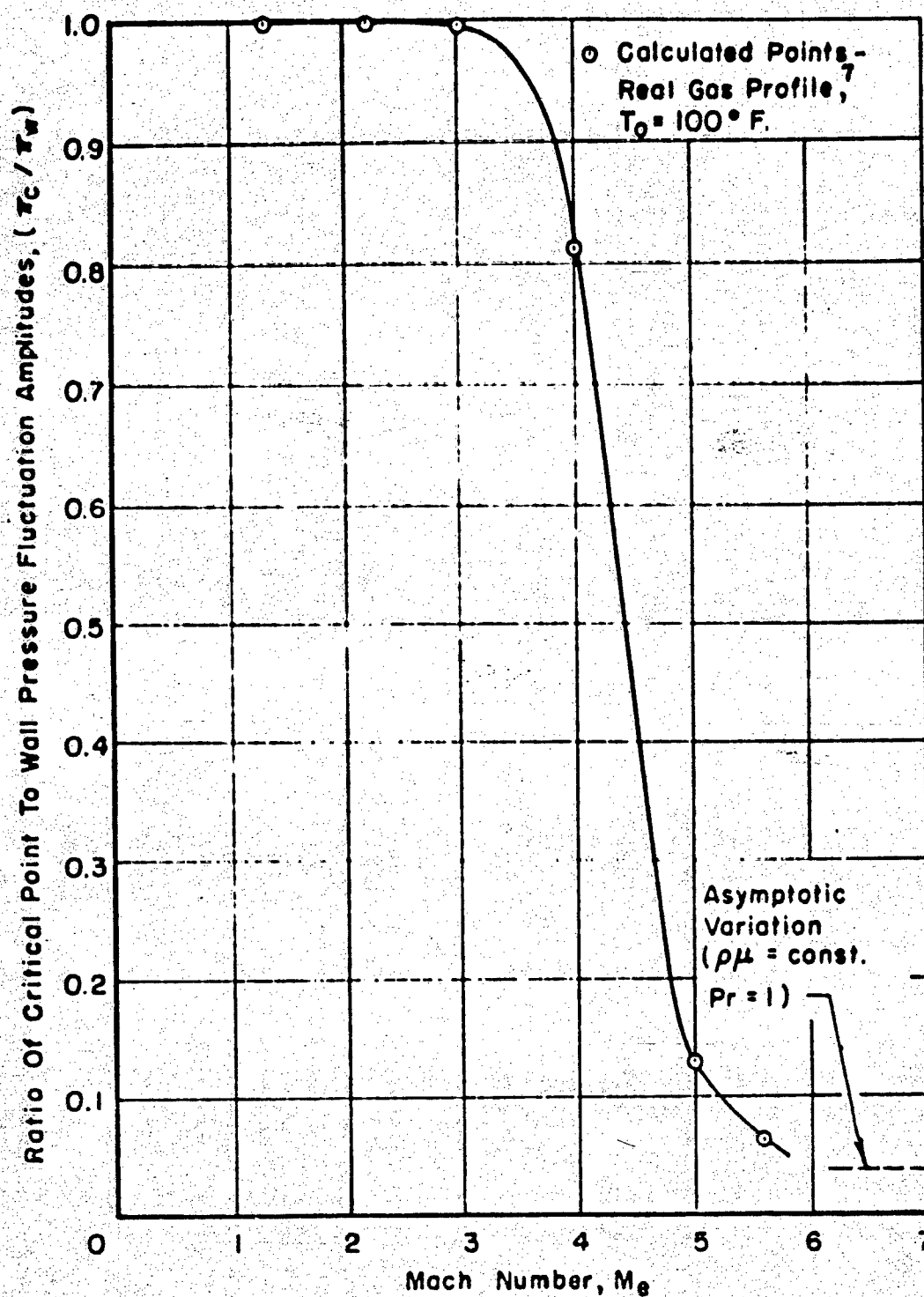


FIG. 3 - PRESSURE FLUCTUATION AMPLITUDE AT CRITICAL POINT,  
NEUTRAL INVISCID OSCILLATION

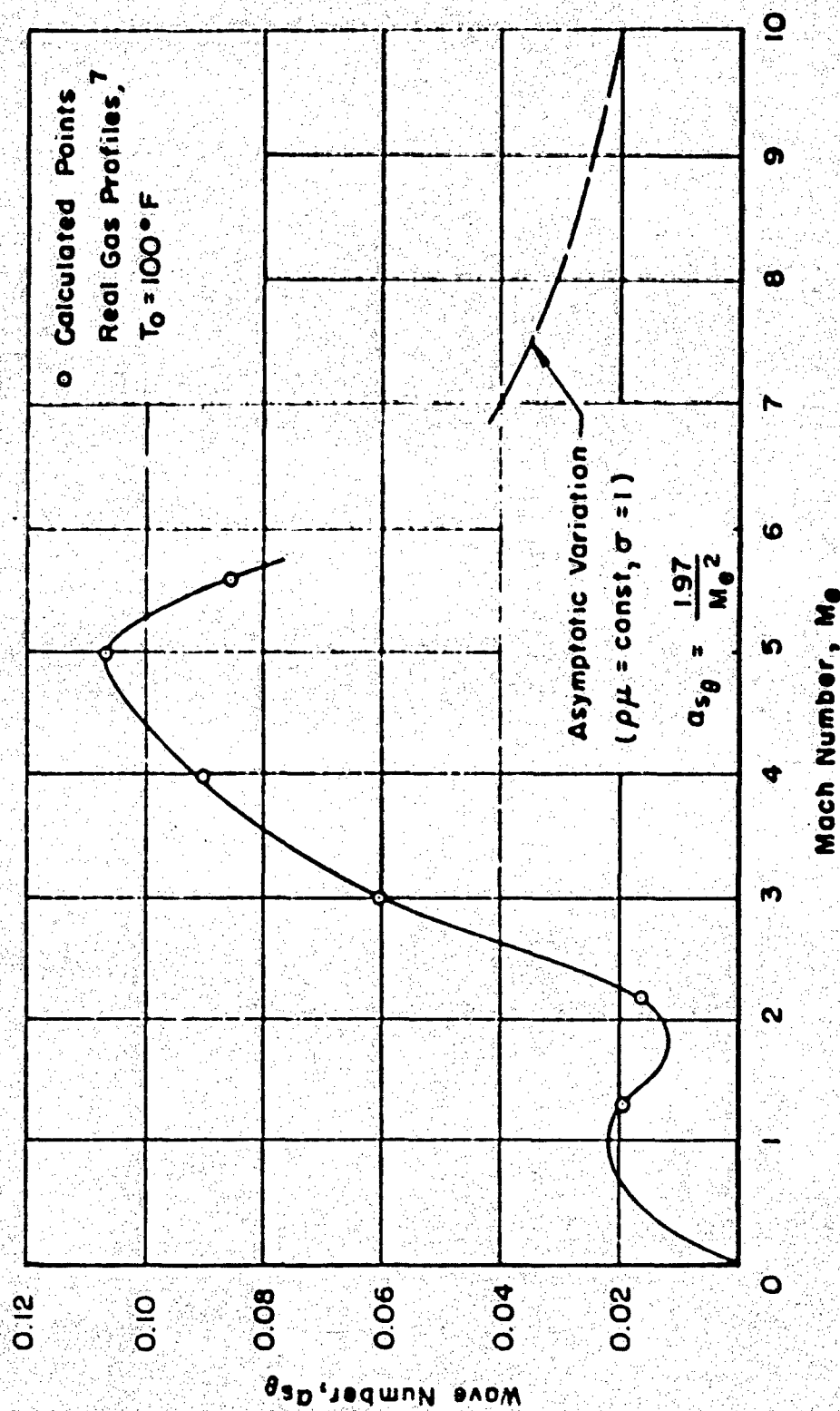
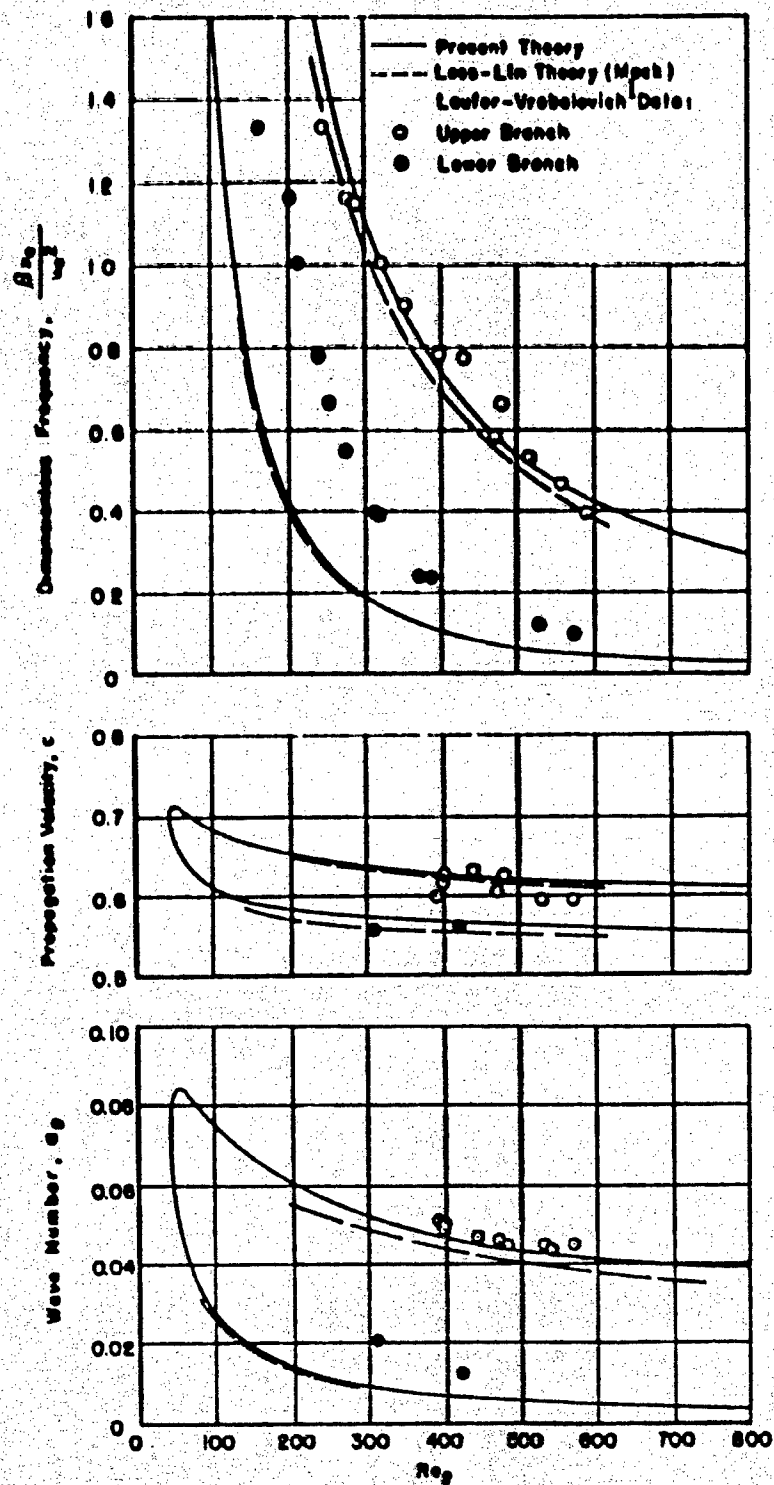


FIG. 4 - WAVE NUMBER OF NEUTRAL INVESED OSCILLATION

FIG. 5 - NEUTRAL STABILITY CHARACTERISTICS,  $M_\infty = 2.2$

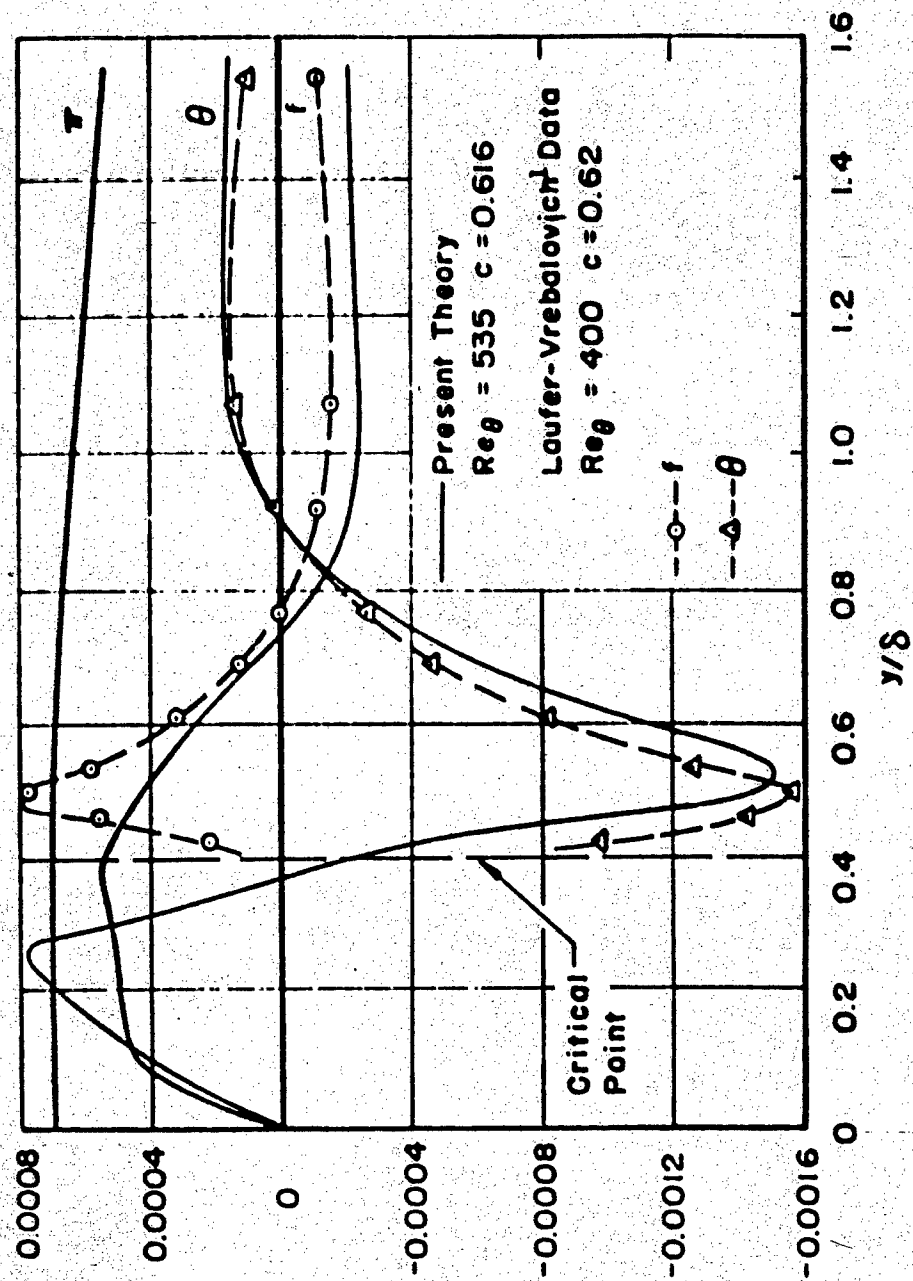


FIG. 6 - AMPLITUDE DISTRIBUTIONS FOR NEUTRAL OSCILLATION,

$Ma = 2.2$

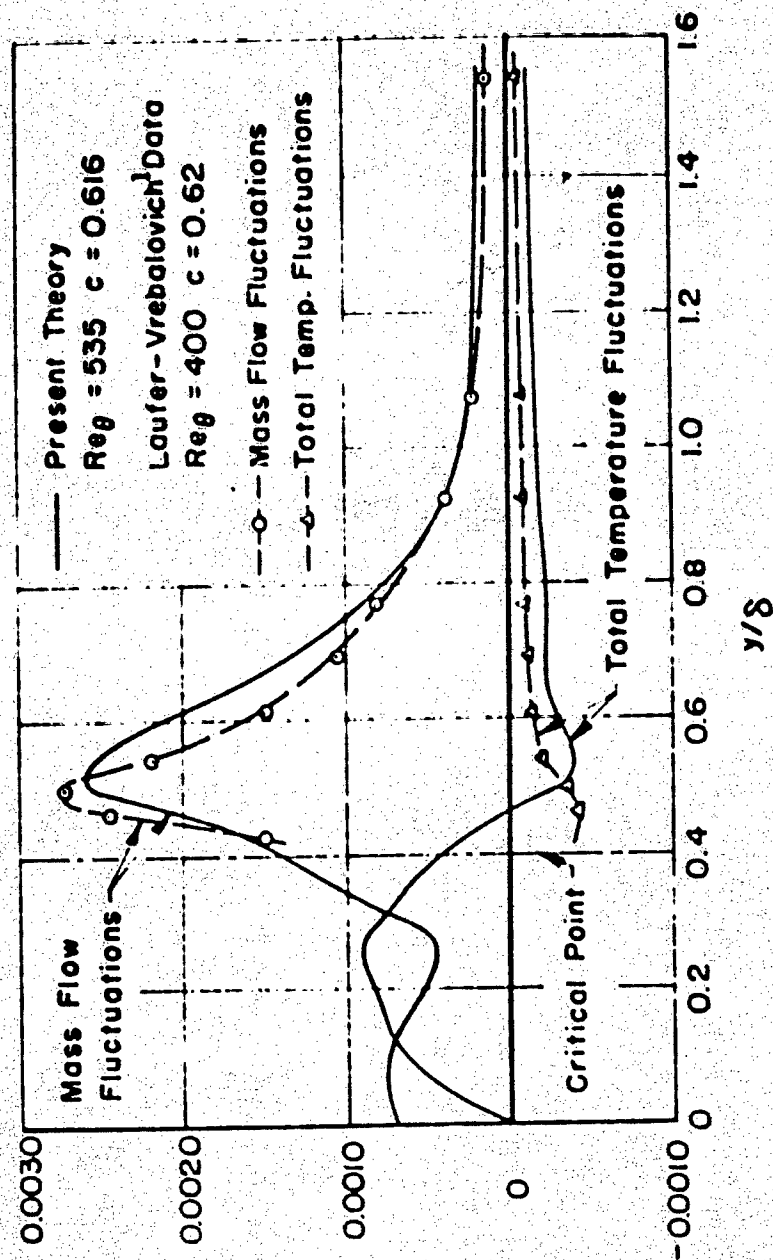


FIG. 7 - MASS FLOW AND TOTAL TEMPERATURE DISTRIBUTIONS FOR NEUTRAL OSCILLATION,  $M_\infty = 2.2$

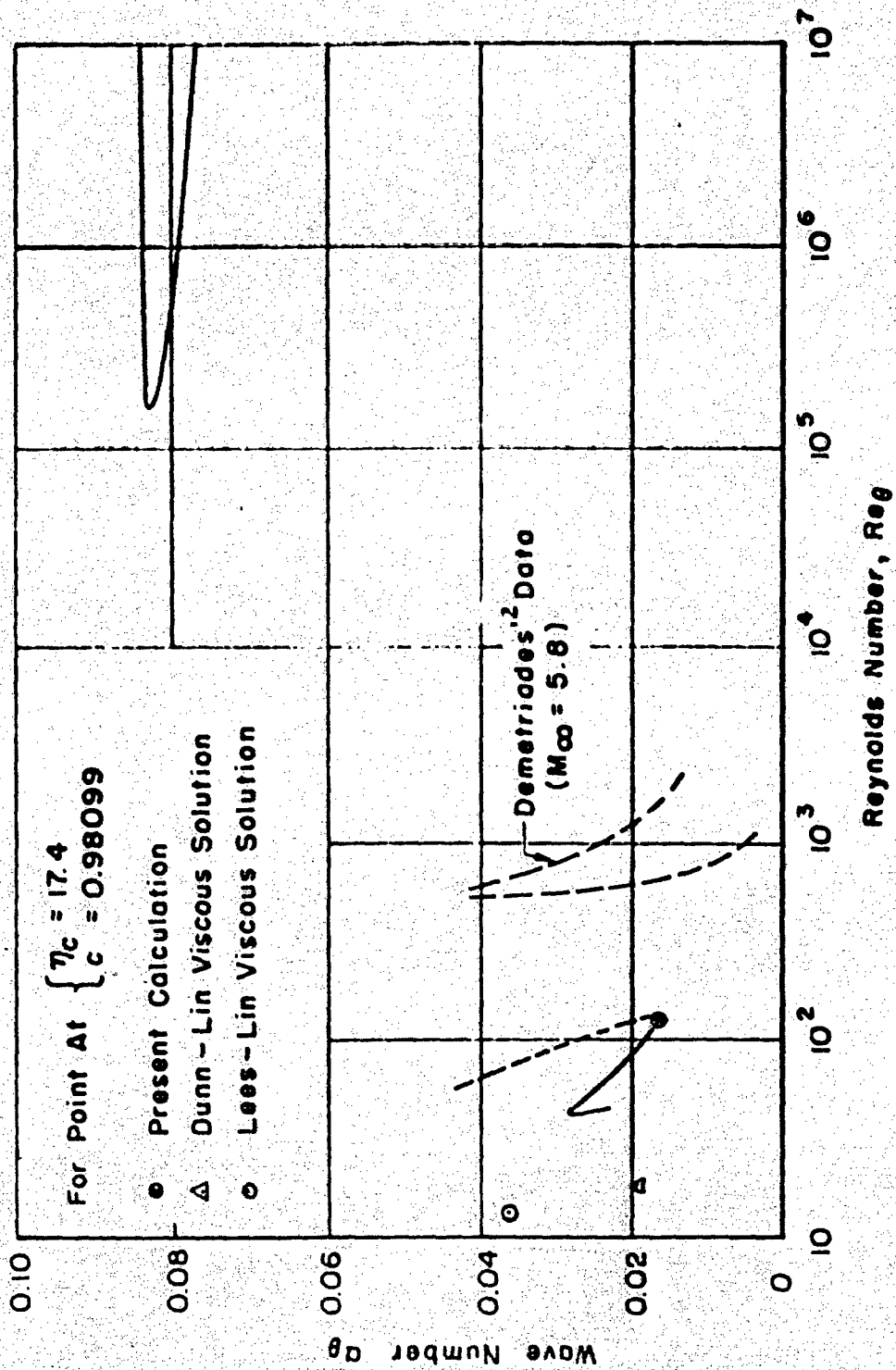


FIG. 8 - NEUTRAL STABILITY CHARACTERISTICS,  $M_\infty = 5.6$